A Novel kind of WVD Associated with the Linear Canonical Transform

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***Abstract*—Linear canonical transformation (LCT), as an extension of the Fourier transform and fractional Fourier transform, has emerged as a useful tool for the nonstationary signals processing. In this paper, a novel kind of Wigner-Ville distribution associated with the LCT (WVL) is presented. First, in order to eliminate the coupling between time variable t and lag variable τ, the WVL is defined based on the scaled parametric symmetric instantaneous autocorrelation function. Second, the WVL of multicomponent linear frequency modulated (LFM) signal is investigated, and the results show that WVL can mitigate the influence of some cross-terms. Finally, through simulation experiments and comparison with the classical WVD, it is verified that WVL has a better ability to suppress cross-terms and significantly improves the detection of multicomponent LFM signals.**

I. Introduction

Linear frequency modulated (LFM) [1] signal is a kind of nonstationary signal widely used in radar, sonar and communication systems, it has an important characteristic that each component can be uniquely determined by the centroid frequency and chirp rate (CFCR) and the spectral components of a LFM signal vary with time. Since traditional Fourier transform (FT) cannot meet the requirement of observation frequency changing with time, the LFM signals are often analyzed using time-frequency (TF) analysis methods. Common TF analysis methods include the short-time Fourier transform (STFT) [2], Wigner-Ville distribution (WVD) [3], short-time Wigner-Ville distribution (STWD) [4], wavelet transform (WT) [5], Wigner-Hough transform (WHT) [6], Radon-Wigner transform (RWT) [7], fractional Fourier transform (FRFT) [8] [9], linear canonical transform (LCT) [10] and time-fractional-frequency (TFF) [11]. Compared with other TF analysis methods, WVD has excellent TF resolution and energy concentration, making it more effective for the analysis of LFM signals. Despite this, when it is applied to multicomponent LFM signals, it is affected by cross-term interference. Therefore, reducing the impact of cross-term (interference) on auto-term (target) has become an important research focus.

In 2009, Lv et al., using the ideal of keystone formatting, proposed the keystone-Wigner transform (KWT) [12], which introduced a weight factor into the time-lag instantaneous autocorrelation function to decouple the time variable t and the lag variable τ. Along with this idea, Lv et al. improved the KWT and proposed the Lv's distribution (LVD) [13], which circumvented all the problems of the KWT, and enhanced the energy of the auto-term meanwhile reduced the energy of the cross-term in 2011. In 2017, the short time Fourier transform-multiple invariance-estimation of signal parameters via rotational invariance techniques (STFT-MI-ESPRIT) [2] was proposed by Cui et al., which improved the estimation precision greatly while maintaining computational complexity. In the same year, Wang et al. proposed the WHT [6], which used the double-deck weight to mitigate the impact of noise and disturbance. In [3] a new method to eliminate cross-terms in the WVD of multicomponent LFM signals was investigated. This method improved the signal to noise ratio (SNR) while maintaining high resolution by removing the cross-terms. In 2021, the definition of WVD associated with LCT (WVDL) [14] was proposed by Xin et al., which achieves smaller errors and lower computational costs compared with other WVD in LCT domain. [15] investigated the four-dimensional Wigner distributions associated with the linear canonical transform (WDLs) for the analytic signals in 2024.

This paper proposes a novel kind of WVD associated with the LCT (WVL). By using a scaling function, it decouples t and τ in the parametric symmetric instantaneous autocorrelation function (PSIAF) [13], eliminating the influence of some cross-terms thereby enhancing the accuracy of detecting multicomponent LFM signals. Then taking a 2D LCT [16] over the new scaled PSIAF time variable and lag variable respectively, can obtain WVL. Compared to the classical WVD, WVL offers more free parameters and greater flexibility.

The remainder of this paper is organized as follows: Section II reviews the relevant theories of LCT. In section III, presents the WVL from a new point of view and derive some properties and theorems. Section IV show the simulation results of the theorems and the effectiveness of the proposed tehniques. Section V is the conclution.

II. Preliminary

A. Two dimentional LCT

The 2D LCT of a signal $f(x,y)$ with parameter $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ is defined as follows[16]

$$
\begin{aligned}
&L_f^{A,B}[f(x,y)] = L_f^{A,B}(u,v)\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)K_{A,B}(x,y,u,v)dxdy\\
&= \begin{cases} \frac{1}{2\pi j}\sqrt{\frac{1}{b_1b_2}}e^{j[(\frac{d_1u^2}{2b_1}+\frac{d_2v^2}{2b_2})]}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\\ \times e^{-j(\frac{ux}{b_1}+\frac{vy}{b_2})}e^{j(\frac{a_1x^2}{2b_1}+\frac{a_2y^2}{2b_2})}dxdy, b_1b_2 \neq 0,\ |A|=|B|=1,\\ \sqrt{d_1d_2}e^{j[(\frac{c_1d_1u^2+c_2d_2v^2}{2})]}f(d_1u,d_2v), b_1^2+b_2^2=0, \end{cases}
\end{aligned} \tag{1}
$$

where $K_{A,B}(x,y,u,v) = K_A(x,u)K_B(y,v)$,

$$
K_A(x,u) = \sqrt{\frac{1}{2\pi jb_1}}e^{j\frac{d_1u^2}{2b_1}}e^{-j\frac{ux}{b_1}}e^{j\frac{a_1x^2}{2b_1}},
$$

$$
K_B(y,v) = \sqrt{\frac{1}{2\pi jb_2}}e^{j\frac{d_2v^2}{2b_2}}e^{-j\frac{vy}{b_2}}e^{j\frac{a_2y^2}{2b_2}}.
$$

Some basic properties of 2D LCT have been studied, this paper uses the following two important properties:
(1) Let $h(x,y) = f(x,y)\times g(x,y)$. $L_h^{A,B}(u,v), L_g^{A,B}(u,v)$ denote 2D LCT of the $h(x,y), g(x,y)$ respectively, while $L_f(u,v)$ denotes 2D FT of the $f(x,y)$. The 2D LCT product theorem is defined as follows [17]

$$
\begin{aligned}
L_h^{A,B}(u,v) &= \frac{1}{4\pi^2}\frac{1}{|b_1b_2|}e^{j(\frac{d_1}{2b_1}u^2+\frac{d_2}{2b_2}v^2)}\\
&\times[(L_g^{A,B}(u,v)e^{-j(\frac{d_1}{2b_1}u^2+\frac{d_2}{2b_2}v^2)}) * L_f(\frac{u}{b_1},\frac{v}{b_2})].
\end{aligned} \tag{2}
$$

(2) Let $g(x,y) = e^{j(xu_0+yv_0)}f(x,y)$. $L_g^{A,B}(u,v), L_f^{A,B}(u,v)$ denote 2D LCT of the $g(x,y), f(x,y)$ respectively. The 2D LCT scaling properties is defined as follows [17]

$$
\begin{aligned}
L_g^{A,B}(u,v) &= e^{j(uu_0d_1+vv_0d_2)-\frac{j}{2}(u_0^2b_1d_1+v_0^2b_2d_2)}\\
&\times L_f^{A,B}(u-u_0b_1, v-v_0b_2),\ u_0,v_0\in R.
\end{aligned} \tag{3}
$$

B. WVD and AF Associated with LCT

The WVD of a signal $f(t)$ associated with the LCT (WDL) with parameter $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ is defined as [18]

$$
\begin{aligned}
W_A^f[f(t)](t,u) &= WVD_{(a,b,c,d)}(t,u)\\
&= \int_{-\infty}^{+\infty} R_f(t,\tau)K_A(u,\tau)d\tau\\
&= \int_{-\infty}^{+\infty} f(t+\frac{\tau}{2})f^*(t-\frac{\tau}{2}))K_A(u,\tau)d\tau,
\end{aligned} \tag{4}
$$

where $K_A(u,\tau) = \sqrt{\frac{1}{j2\pi b}}e^{j\frac{d}{2b}u^2}e^{-j\frac{u}{b}\tau}e^{j\frac{a}{2b}\tau^2}$, $|A|=1$.

The AF of a signal $f(t)$ associated with the LCT (AFL) with parameter $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ is defined as [19]

$$
\begin{aligned}
AFL[f(t)](\tau,u) &= AF_{(a,b,c,d)}(\tau,u)\\
&= \int_{-\infty}^{+\infty} R_f(t,\tau)K_A(t,u)dt\\
&= \int_{-\infty}^{+\infty} f(t+\frac{\tau}{2})f^*(t-\frac{\tau}{2})K_A(t,u)dt,
\end{aligned} \tag{5}
$$

where $K_A(t,u) = \sqrt{\frac{1}{j2\pi b}}e^{j\frac{d}{2b}u^2}e^{-j\frac{u}{b}t}e^{j\frac{a}{2b}t^2}$.

III. Main Results

Based on the aforementioned research of the TF analysis methods, we propose a novel kind of WVD associated with the LCT in this section.

A. Definition

Definition 1. *Suppose the kernel of the 2D LCT with parameter A, B is $K_{A,B}$, then the WVL of a signal $x(t)$ is defined as*

$$
\begin{aligned}
WVL_x^{A,B}(u,v) &= L_\Gamma^{A,B}(\Gamma[R_x^C(t,\tau)])\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma[R_x^C(t,\tau)]K_{A,B}(t_n,\tau,u,v)dt_nd\tau\\
&= \sum_{i=0}^{K-1} WVL_{x_i}(u,v) + \sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1} WVL_{x_ix_j}(u,v),
\end{aligned} \tag{6}
$$

where $A = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$, and

$$
\Gamma[G(t,\tau)] \to G(\tfrac{t_n}{h(\tau+m)},\tau)
$$

$$
R_x^C(t,\tau) = x(t+\tfrac{\tau+m}{2})x^*(t-\tfrac{\tau+m}{2}),
$$

where Γ is a scale operator and $R_x^C(t,\tau)$ is the PSIAF of $x(t)$.

B. Properties

Property 1. *Conjugation symmetry property*
Suppose the WVL of a signal $x(t)$ is denoted as $WVL_x^{A,B}(u,v)$, the WVL of $x^*(t)$ is

$$
WVL_{x^*}^{A,B}(u,v) = [WVL_x^{A^{-1},B^{-1}}(u,v)]^*. \tag{7}
$$

Property 2. *Shifting property*
Suppose the WVL of a signal $x(t)$ is denoted as $WVL_x^{A,B}(u,v)$, the WVL of $\widetilde{x}(t) = x(t)e^{jwt}$ is

$$
\begin{aligned}
&WVL_{\widetilde{x}}^{A,B}(u,v)\\
&= e^{j(wa+d_2wv-\frac{b_2d_2w^2}{2})}WVL_x^{A,B}(u,v-wb_2).
\end{aligned} \tag{8}
$$

C. Theorems

Consider an infinite-time multicomponent LFM signal with the constant amplitude A_i, the centroid frequency f_i and chirp rate γ_i as

$$
x(t) = \sum_{i=0}^{K-1} x_i(t) = \sum_{i=0}^{K-1} A_ie^{j2\pi f_it+j\pi\gamma_it^2}, \tag{9}
$$

where K is the number of signal components present in the signal. The PSIAF of $x(t)$ is defined by[13]

$$
\begin{aligned}
R_x^C(t,\tau) &= x(t+\frac{\tau+m}{2})x^*(t-\frac{\tau+m}{2})\\
&= \sum_{i=0}^{K-1} A_i^2e^{j2\pi f_i(\tau+m)}e^{j2\pi\gamma_i(\tau+m)t}\\
&+ \sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}(R_{x_ix_j}^C(t,\tau)+R_{x_jx_i}^C(t,\tau)),
\end{aligned} \tag{10}
$$

where $R^C_{x_ix_j}(t,\tau) = A_iA_je^{j\pi[\frac{\gamma_i-\gamma_j}{4}(\tau+m)^2+(f_i+f_j)(\tau+m)]}$ $\times e^{j\pi[(\gamma_i-\gamma_j)t^2+(2(f_i-f_j)+(\gamma_i+\gamma_j)(\tau+m))t]}$.
Because of the lag variable τ and time variable t couple with each other in the exponential phase terms, we use the scaling operator Γ of LVD [13] on the function $R^C_x(t,\tau)$. Then we can get $\Gamma[R^C_x(t,\tau)]$ as

$$\begin{aligned}\Gamma[R^C_x(t,\tau)] =& \sum_{i=0}^{K-1} A_i^2 e^{(j2\pi f_i(\tau+m)+j2\pi\frac{\gamma_i}{h}t_n)}\\ &+\sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}\Gamma[R^C_{x_ix_j}(t_n,\tau)+R^C_{x_jx_i}(t_n,\tau)],\end{aligned} \tag{11}$$

where $\Gamma[R^C_{x_ix_j}(t_n,\tau)+R^C_{x_jx_i}(t_n,\tau)] = A_iA_jR^{C_1}_{x_ix_j}(t_n,\tau)$ $\times R^{C_2}_{x_ix_j}(t_n,\tau)$ and

$$\begin{aligned}&R^{C_1}_{x_ix_j}(t_n,\tau) = e^{j\pi[(\gamma_i+\gamma_j)\frac{t_n}{h}+(f_i+f_j)(\tau+m)]},\\ &R^{C_2}_{x_ix_j}(t_n,\tau)\\ =&e^{j\pi(2(f_i-f_j)\frac{t_n}{h(\tau+m)}+(\gamma_i-\gamma_j)\frac{t_n^2}{h^2(\tau+m)^2}+\frac{\gamma_i-\gamma_j}{4}(\tau+m)^2)}\\ =&2cos(\pi[(\gamma_i-\gamma_j)\frac{t_n^2}{h^2(\tau+m)^2}+2(f_i-f_j)\frac{t_n}{h(\tau+m)}\\ &+\frac{(\tau+m)^2}{4}(\gamma_i-\gamma_j)]).\end{aligned} \tag{12}$$

As shown in [11], the scaled PSIAF of $x(t)$ can be expressed as two parts: auto-terms and cross-terms. Different from the existing methods [13], we take place of the kernel of 2D FT with the kernel of 2D LCT to get a novel kind of WVD associated with the LCT (WVL). The result is divided into two parts, auto-terms and cross-terms, leading to the following interesting theorems.

Theorem 1. *Each auto term in the WVL with parameter $a_1 = a_2 = 0$ can be expressed as*

$$\begin{aligned}WVL^{A,B}_{x_i}(u,v) =& \frac{A_i^2}{2\pi j}\sqrt{\frac{1}{b_1b_2}}e^{j(\frac{d_1u^2}{2b_1}+\frac{d_2v^2}{2b_2})}e^{j2\pi f_im}\\ &\times\delta(\frac{u}{2\pi b_1}-\frac{\gamma_i}{h})\delta(\frac{v}{2\pi b_2}-f_i).\end{aligned} \tag{13}$$

Proof: $WVL^{A,B}_{x_i}(u,v)$

$$\begin{aligned}=&\frac{A_i^2}{2\pi j}\sqrt{\frac{1}{b_1b_2}}e^{j(\frac{d_1u^2}{2b_1}+\frac{d_2v^2}{2b_2})}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{j(\frac{a_1t_n^2}{2b_1}+\frac{a_2\tau^2}{2b_2})}\\ &\times e^{-j(\frac{ut_n}{b_1}+\frac{v\tau}{b_2})}e^{j2\pi f_i(\tau+m)+j2\pi\gamma_i\frac{t_n}{h}}dt_nd\tau\\ =&\frac{A_i^2}{2\pi j}\sqrt{\frac{1}{b_1b_2}}e^{j(\frac{d_1u^2}{2b_1}+\frac{d_2v^2}{2b_2})}e^{j2\pi f_0m}\delta(\frac{u}{2\pi b_1}-\frac{\gamma_i}{h})\delta(\frac{v}{2\pi b_2}-f_i).\end{aligned}$$

■

It is seen from Theorem 1 that the auto-term of WVL has the characteristics of energy accumulation.providing a solid foundation for further signal processing. Next, we explore the theorem 2 associated with the cross-terms.

Theorem 2. *Each cross term in the WVL with parameter $a_1 = a_2 = 0$, can be modeled as following cases and let $\widetilde{u} = \frac{uh}{2\pi b_1} - \frac{\gamma_i+\gamma_j}{2}, \widetilde{v} = \frac{v}{2\pi b_2} - \frac{f_i+f_j}{2}$.*

Case 1. For $\gamma_i = \gamma_j \neq \frac{uh}{2\pi b_1}, f_i \neq f_j$,

$$\begin{aligned}&WVL^{A,B}_{x_ix_j}(u,v)\\ =&\frac{A_iA_jh}{4j\pi^3|b_1b_2|}\sqrt{\frac{1}{b_1b_2}}e^{j(\frac{d_1}{2b_1}u^2+\frac{d_2}{2b_2}v^2)}\\ &\times e^{j\pi(f_i+f_j)m}e^{j2\pi m\widetilde{v}}\frac{f_i-f_j}{\widetilde{u}^2}cos(\frac{2\pi(f_i-f_j)}{\widetilde{u}}\widetilde{v}).\end{aligned} \tag{14}$$

Case 2. For $\gamma_i = \gamma_j = \frac{uh}{2\pi b_1}, f_i \neq f_j$,

$$WVL^{A,B}_{x_ix_j}(u,v) = 0. \tag{15}$$

Case 3. For $\gamma_i \neq \gamma_j,\ u = \pm|\frac{\pi(\gamma_i-\gamma_j)b_1}{h}|$,

$$\begin{aligned}&WVL^{A,B}_{x_ix_j}(u,v)\\ =&\frac{A_iA_j|h||\gamma_i-\gamma_j|^{\frac{3}{2}}}{j2\pi^3\sqrt{b_1b_2}[(\gamma_i-\gamma_j)\widetilde{v}-(f_i-f_j)\widetilde{u}]^2}e^{j2\pi m\widetilde{v}}\\ &\times e^{j\pi[\frac{(\gamma_i+\gamma_j)d_1}{h}u+(f_i+f_j)d_2v]}cos(\frac{\pi^2b_1d_1}{h^2}[2\widetilde{u}^2+\frac{(\gamma_i+\gamma_j)^2}{2}]\\ &+\pi^2b_2d_2[2\widetilde{v}^2+\frac{(f_i+f_j)^2}{2}]-\frac{\pi(f_i-f_j)^2}{\gamma_i-\gamma_j}-sgn(\frac{\gamma_i-\gamma_j}{2})\frac{\pi}{4}).\end{aligned} \tag{16}$$

Case 4. For $\gamma_i \neq \gamma_j, u \neq \pm|\frac{\pi(\gamma_i-\gamma_j)b_1}{h}|, \widetilde{u} \neq \pm|\frac{(\gamma_i-\gamma_j)}{2}|$,

$$\begin{aligned}&WVL^{A,B}_{x_ix_j}(u,v)\\ =&-\frac{8A_iA_jh[(\gamma_i-\gamma_j)\widetilde{v}-(f_i-f_j)\widetilde{u}]}{\pi\sqrt{b_1b_2}[(\gamma_i-\gamma_j)^2-4\widetilde{u}^2]^{\frac{3}{2}}}e^{j\pi[\frac{(\gamma_i+\gamma_j)d_1}{h}u+(f_i+f_j)d_2v]}\\ &\times sin(\frac{\pi^2b_1d_1}{h^2}[\frac{(\gamma_i+\gamma_j)^2}{2}-2\widetilde{u}^2]+\pi^2b_2d_2[\frac{(f_i+f_j)^2}{2}-2\widetilde{v}^2]\\ &+\pi\frac{(f_i-f_j)^2}{\gamma_i-\gamma_j}+\frac{4\pi[(\gamma_i-\gamma_j)\widetilde{v}-(f_i-f_j)\widetilde{u}]^2}{(\gamma_i-\gamma_j)[(\gamma_i-\gamma_j)^2-4\widetilde{u}^2]}\\ &-\frac{\pi}{4}sgn(\frac{\gamma_i-\gamma_j}{2})(1+sgn(\pi^2b_1^2[(\gamma_i-\gamma_j)-4\widetilde{u}^2]))).\end{aligned} \tag{17}$$

Case 5. For $\gamma_i \neq \gamma_j, u \neq \pm|\frac{\pi(\gamma_i-\gamma_j)b_1}{h}|, \widetilde{u} = \pm|\frac{(\gamma_i-\gamma_j)}{2}|$,

$$WVL^{A,B}_{x_ix_j}(u,v) = 0. \tag{18}$$

Proof: For $\gamma_i = \gamma_j$, using (2), we have

$$\begin{aligned}&WVL^{A,B}_{x_ix_j}(u,v)\\ =&L^{t_n,\tau}\{2A_iA_jR^{C_1}_{x_ix_j}(t_n,\tau)cos(\pi[2(f_i-f_j)\frac{t_n}{h(\tau+m)}])\}\\ =&\frac{1}{4\pi^2}\frac{1}{|b_1b_2|}e^{j(\frac{d_1}{2b_1}u^2+\frac{d_2}{2b_2}v^2)}[L^{t_n,\tau}\{2A_iA_jR^{C_1}_{x_ix_j}(t_n,\tau)\}\\ &\times e^{-j(\frac{d_1}{2b_1}u^2+\frac{d_2}{2b_2}v^2)}]*F^{t_n,\tau}(cos(\pi[2(f_i-f_j)\frac{t_n}{h(\tau+m)}])).\end{aligned} \tag{19}$$

First, taking 2D LCT into $2A_iA_jR^{C_1}_{x_ix_j}(t_n,\tau)$, we have

$$\begin{aligned}&L^{t_n,\tau}\{2A_iA_jR^{C_1}_{x_ix_j}(t_n,\tau)\}\\ =&\frac{A_iA_j}{\pi j}\sqrt{\frac{1}{b_1b_2}}e^{j[\frac{d_1u^2}{2b_1}+\frac{d_2v^2}{2b_2}]+j\pi(f_i+f_j)m}\\ &\times\delta(\frac{u}{2\pi b_1}-\frac{\gamma_i}{h})\delta(\frac{v}{2\pi b_2}-\frac{f_i+f_j}{2}).\end{aligned} \tag{20}$$

Second, taking 2D LCT into $cos(\pi[2(f_i - f_j)\frac{t_n}{h(\tau+m)}])$, we have

$$F^{t_n,\tau}(cos(\pi[2(f_i - f_j)\frac{t_n}{h(\tau+m)}])) = \begin{cases} \frac{f_i - f_j}{hu^2} e^{j2\pi mv} cos(\frac{2\pi(f_i - f_j)}{hu}v), f_i \neq f_j, \\ \delta(v)\delta(u), f_i = f_j. \end{cases} \tag{21}$$

Then take (20) and (21) into (19). Because of $\widetilde{u} = \frac{uh}{2\pi b_1} - \frac{\gamma_i+\gamma_j}{2}, \widetilde{v} = \frac{v}{2\pi b_2} - \frac{f_i+f_j}{2}$, we can obtain (14) and (15).

For $\gamma_i \neq \gamma_j$, taking a LCT of $R^{C_2}_{x_i x_j}(t_n,\tau)$ with respect to t_n, we have

$$\begin{aligned} &L^{t_n}(R^{C_2}_{x_i x_j}(t_n,\tau)) \\ =&\sqrt{\frac{1}{j2\pi b}} e^{j\frac{d}{2b}u^2} e^{j\frac{\pi(\gamma_i-\gamma_j)}{4}(\tau+m)^2} \\ &\times \int_{-\infty}^{+\infty} e^{j(\frac{\pi(\gamma_i-\gamma_j)}{h^2(\tau+m)^2}t_n^2 + [\frac{2\pi(f_i-f_j)}{h(\tau+m)} - \frac{u}{b}]t_n)} dt_n. \end{aligned} \tag{22}$$

Using the principle of stationary phase (PSP) [20], first we let

$$\frac{d}{dt_n}(\text{phase}(R^{C_2}_{x_i x_j}(t_n,\tau)) - 2\pi\gamma t_n) = 0, \tag{23}$$

we have

$$t_n^* = (\frac{u}{b} - \frac{2\pi(f_i - f_j)}{h(\tau+m)})\frac{h^2(\tau+m)^2}{2\pi(\gamma_i - \gamma_j)}. \tag{24}$$

Second

$$\frac{d^2}{dt_n^2}(\text{phase}(R^{C_2}_{x_i x_j}(t_n^*,\tau)) - 2\pi\frac{u}{b}t_n^*) = \frac{2\pi(\gamma_i - \gamma_j)}{h^2(\tau+m)^2}. \tag{25}$$

Then

$$\begin{aligned} &L^{t_n}(R^{C_2}_{x_i x_j}(t_n,\tau)) \\ =&\sqrt{\frac{1}{j2\pi b}} e^{j\frac{d}{2b}u^2} \frac{|h(\tau+m)|}{\sqrt{\gamma_i - \gamma_j}} e^{jsgn(\frac{\gamma_i-\gamma_j}{2})\frac{\pi}{4}} \\ &\times e^{j[-\frac{h^2(\tau+m)^2}{4\pi(\gamma_i-\gamma_j)b^2}u^2 + \frac{h(\tau+m)(f_i-f_j)}{(\gamma_i-\gamma_j)b}u - \frac{\pi(f_i-f_j)^2}{\gamma_i-\gamma_j} + \frac{\pi(\gamma_i-\gamma_j)}{4}(\tau+m)^2]}. \end{aligned} \tag{26}$$

For $u = \pm|\frac{\pi(\gamma_i-\gamma_j)b_1}{h}|$, applying $\mathcal{F}(|(\tau+m)|) = -\frac{1}{2\pi^2 f^2}$ [3], we have

$$\begin{aligned} &WVL'_{x_i x_j}(u,v) = L^{\tau}(L^{t_n}(R^{C_2}_{x_i x_j}(t_n,\tau))) \\ =&\frac{1}{j2\pi}\frac{1}{\sqrt{b_1 b_2}}\frac{|h|}{\sqrt{|\gamma_i - \gamma_j|}} e^{-j(\frac{d_1}{2b_1}u^2 + \frac{d_2}{2b_2}v^2)} e^{j\frac{\pi(f_i-f_j)^2}{\gamma_i-\gamma_j}} e^{jsgn(\frac{\gamma_i-\gamma_j}{2})\frac{\pi}{4}} \\ &\times e^{j\frac{v}{b_2}m} \int_{-\infty}^{+\infty} |(\tau+m)| e^{-j2\pi(\frac{v}{2\pi b_2} - \frac{h(f_i-f_j)u}{2\pi(\gamma_i-\gamma_j)b_1})(\tau+m))} d(\tau+m) \\ =&\frac{1}{j2\pi}\frac{1}{\sqrt{b_1 b_2}} e^{-j(\frac{d_1}{2b_1}u^2 + \frac{d_2}{2b_2}v^2)} e^{j\frac{\pi(f_i-f_j)^2}{\gamma_i-\gamma_j}} e^{jsgn(\frac{\gamma_i-\gamma_j}{2})\frac{\pi}{4}} e^{j\frac{v}{b_2}m} \\ &\times \frac{|h|}{\sqrt{|\gamma_i - \gamma_j|}} \frac{2}{(\frac{v}{b_2} - \frac{h(f_i-f_j)u}{(\gamma_i-\gamma_j)b_1})^2}. \end{aligned} \tag{27}$$

Denote * by the complex conjugate. Using the conjugation property of the LCT, we obtain

$$\begin{aligned} &L^{\tau}(L^{t_n}(R^{C_2^*}_{x_i x_j}(t_n,\tau))) = L^{\tau}(L^{t_n^*}(\tau,-u)) \\ =&WVL'^*_{x_i x_j}(-v,-u). \end{aligned} \tag{28}$$

Using (3), we have

$$\begin{aligned} &L^{t_n,\tau}(R^{C_1}_{x_i x_j}(t_n,\tau)R^{C_2}_{x_i x_j}(t_n,\tau)) \\ =&e^{j(u\frac{\pi(\gamma_i+\gamma_j)}{h}d_1 + v\pi(f_i+f_j)d_2) - \frac{j}{2}[(\frac{\pi(\gamma_i+\gamma_j)}{h})^2 b_1 d_1 + \pi^2(f_i+f_j)^2 b_2 d_2]} \\ &\times L^{t_n,\tau}(R^{C_2}_{x_i x_j}(u - \frac{\pi(\gamma_i+\gamma_j)}{h}b_1, v - \pi(f_i+f_j)b_2)). \end{aligned} \tag{29}$$

Obviously

$$\begin{aligned} &L^{t_n,\tau}_{\Gamma}(A_i A_j R^{C_1}_{x_i x_j}(t_n,\tau)[R^{C_2}_{x_i x_j}(t_n,\tau) + R^{C_2^*}_{x_i x_j}(t_n,\tau)]) \\ =&A_i A_j[L^{t_n,\tau}(R^{C_1}_{x_i x_j}(t_n,\tau)R^{C_2}_{x_i x_j}(t_n,\tau)) \\ &+ L^{t_n,\tau}(R^{C_1}_{x_i x_j}(t_n,\tau)R^{C_2^*}_{x_i x_j}(t_n,\tau))], \end{aligned} \tag{30}$$

substituting $\widetilde{u}$ and $\widetilde{v}$, we have

$$\begin{aligned} &WVL^{A,B}_{x_i x_j}(u,v) \\ =&\frac{A_i A_j |h|}{j2\pi^3\sqrt{b_1 b_2 |(\gamma_i - \gamma_j)|}} \frac{1}{(\widetilde{v} - \frac{(f_i-f_j)\widetilde{u}}{\gamma_i-\gamma_j})^2} e^{j2\pi m\widetilde{v}} \\ &\times e^{j\pi[\frac{(\gamma_i+\gamma_j)d_1}{h}u + (f_i+f_j)d_2 v]} cos(\frac{2\pi^2 b_1 d_1}{h^2}\widetilde{u}^2 + 2\pi^2 b_2 d_2 \widetilde{v}^2 \\ &- \frac{\pi(f_i - f_j)^2}{\gamma_i - \gamma_j} - sgn(\frac{\gamma_i - \gamma_j}{2})\frac{\pi}{4} \\ &+ \frac{\pi^2}{2}[\frac{(\gamma_i+\gamma_j)^2 b_1 d_1}{h^2} + (f_i+f_j)^2 b_2 d_2]). \end{aligned} \tag{31}$$

Simplify (31) we can obtain (16).

For $u \neq \pm|\frac{\pi(\gamma_i-\gamma_j)b_1}{h}|$, we have

$$\begin{aligned} &L^{\tau}(L^{t_n}(R^{C_2}_{x_i x_j}(t_n,\tau))) \\ =&\frac{1}{j2\pi}\frac{1}{\sqrt{b_1 b_2}}\frac{|h|}{\sqrt{|\gamma_i - \gamma_j|}} e^{-j(\frac{d_1}{2b_1}u^2 + \frac{d_2}{2b_2}v^2)} \\ &\times e^{-j\frac{\pi(f_i-f_j)^2}{\gamma_i-\gamma_j}} e^{jsgn(\frac{\gamma_i-\gamma_j}{2})\frac{\pi}{4}} \int_{-\infty}^{+\infty} |(\tau+m)| \\ &\times e^{j[(\frac{\pi(\gamma_i-\gamma_j)}{4} - \frac{h^2u^2}{4\pi(\gamma_i-\gamma_j)b_1^2})(\tau+m)^2 + (\frac{h(f_i-f_j)}{b_1(\gamma_i-\gamma_j)} - \frac{v}{b_2})(\tau+m)]} d(\tau+m). \end{aligned} \tag{32}$$

Similarly, using the PSP, we have

$$(\tau+m)^* = \frac{2\pi((\gamma_i - \gamma_j)b_1^2 v - h(f_i - f_j)b_1 b_2)}{b_2(\pi^2(\gamma_i - \gamma_j)^2 b_1^2 - h^2u^2)}, \tag{33}$$

then

$$\begin{aligned} &WVL'_{x_i x_j}(u,v) \\ =&\frac{2\pi b_1^{\frac{3}{2}} h[b_1(\gamma_i - \gamma_j)v - h(f_i - f_j)b_2 u]}{jb_2^{\frac{3}{2}}[\pi^2(\gamma_i - \gamma_j)^2 b_1^2 - h^2u^2]^{\frac{3}{2}}} \\ &\times e^{j(\frac{d_1}{2b_1}u^2 + \frac{d_2}{2b_2}v^2)} e^{-j\pi\frac{(b_1(\gamma_i-\gamma_j)v - h(f_i-f_j)b_2 u)^2}{b_2^2(\gamma_i-\gamma_j)(\pi^2(\gamma_i-\gamma_j)^2 b_1^2 - h^2u^2)}} \\ &\times e^{j\frac{\pi}{4}sgn(\frac{\gamma_i-\gamma_j}{2})(1+sgn(\pi^2 b_1^2(\gamma_i-\gamma_j) - h^2u^2))} e^{-j\frac{\pi(f_i-f_j)^2}{\gamma_i-\gamma_j}}. \end{aligned} \tag{34}$$

(17) and (18) are derived by substituting (34) into (29) and then applying the (28) to calculate (30). This completes the proof. ■

Theorem 2 shows that WVL can eliminate the effect of some cross terms. When $\gamma_i = \gamma_j = \frac{uh}{2\pi b_1}, f_i \neq f_j$ and $\gamma_i \neq \gamma_j, u \neq \pm|\frac{\pi(\gamma_i - \gamma_j)b_1}{h}|, \widetilde{u} = \pm|\frac{(\gamma_i-\gamma_j)}{2}|$, it is shown from (15) or (18) that the corresponding cross term turns out to be zero.

IV. Simulation

This section proforms the simulation to verify the derived results, and demonstrates the benefits of the WVL in multi-component LFM signal detection.

Firstly, consider a multicomponent LFM signal $x(t)$, which has three components, and the chirp rates are $\gamma_0 = 10, \gamma_1 = 10, \gamma_2 = -20$ respectively, the centroid frequencies are $f_0 = -15, f_1 = 5, f_2 = 5$ respectively. $A_0 = A_1 = A_2 = 1$, then

$$x(t) = e^{j2\pi(-15)t + j\pi 10t^2} + e^{j2\pi 5t + j\pi 10t^2} + e^{j2\pi 5t + j\pi(-20)t^2}. \tag{35}$$

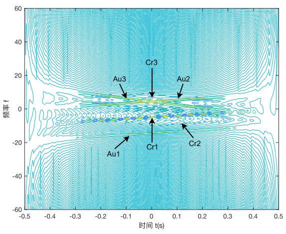

Fig. 1. Classical WVD

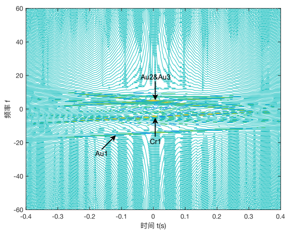

Fig. 2. LCT for the scaled PSIAF

Fig. 1 shows the result of the classical WVD of $x(t)$, Fig. 2 shows the result of the LCT with parameter [0,1/2,2,0] for the scaled PSIAF of $x(t)$. As can be seen from the Fig. 1 and Fig. 2, the instantaneous frequency of each component of the LFM signal varies with time, and the chirp frequency is shown by the slope, and the central frequency is the frequency value at $t = 0$. After taking the scaling operation for the PSIAF, the slope of the auto-terms become zero. Because of $\gamma_0 = \gamma_1$, the Cr1 still exists, however since $\gamma_0 \neq \gamma_2$ and $\gamma_1 \neq \gamma_2$, Cr2 and Cr3 oscillate according to the cosine function respectively, as can be clearly seen from formulas (12).

Secondly, by taking LCT with parameter [0,1/2,2,0] of the Fig. 2 with repect to τ, we can obtain the WVL's result, which is depicted in Fig. 3. When applying the WVL to the multicomponent LFM signal $x(t)$, it can be observed that the auto-terms are delta functions, indicating that WVL exhibits the characteristic of auto-terms energy accumulation. The energy of the cross-terms is negligible compared to the peaks of the auto-terms.

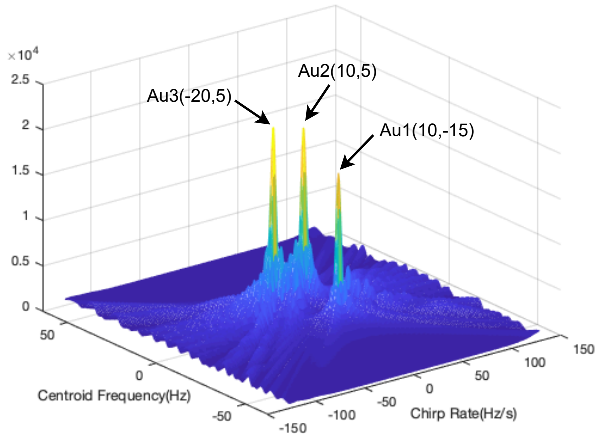

Fig. 3. WVL of multicomponent LFM signal

V. Conclution

Combining the advantages of the WVD and the LCT in the TF domain, this paper proposes the WVL and explores its conjugation symmetry and shifting properties. Especially in processing multicomponent LFM signals, WVL has the advantages of concentrating auto-term energy and eliminating some cross-terms. First, by scaling the PSIAF, the coupling between time variable and lag variable is eliminated. Then, by performing a 2D LCT on t and τ, each auto-term component of the LFM signal can be easily identified in the WVL plane through peak detection. Additionally, the frequency parameters can be directly extracted from the coordinate values, and simulations have verified these conclusions. Therefore, the proposed WVL in this paper has important role for detecting and processing multicomponent LFM signals.

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