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A Discrete-Valued Signal Estimation by Nonconvex Enhancement of SOAV with cLiGME Model

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***Abstract*—In this paper, for the discrete-valued signal estimation, we propose a regularized least squares model but with a nonconvex enhancement of the so-called SOAV convex regularizer. To design more contrastive regularizers whose minima correspond to desired discrete values, we propose a class of nonconvex functions with Generalized Moreau Enhancement (GME) of the weighted ℓ_1-norm. Promisingly, by tuning properly the design parameters of the proposed GME regularizers, (i) we can make the nonconvexly-regularized least squares model convex; and (ii) we can use an iterative algorithm for finding a global minimizer of the proposed model. We also propose a pair of simple technical improvements, of the proposed algorithm, called respectively a generalized superiorization and an iterative reweighting. Numerical experiments demonstrate the effectiveness of the proposed model and algorithms in a scenario of MIMO signal detection.**

I. Introduction

Many tasks in signal processing, including digital communication and discrete-valued control [1]–[7], have been formulated as the following discrete-valued estimation problem:

Problem 1 (A discrete-valued estimation problem)**.**

$$\text{Find } \mathbf{x}^\star \in \mathfrak{D} \text{ such that } \mathbf{y} = \mathbf{A}\mathbf{x}^\star + \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathfrak{D} := \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{L^N}\} = \mathfrak{A}^N := \{a_1, a_2, \ldots, a_L\}^N \subset \mathbb{R}^N$, $\mathbf{y} \in \mathbb{R}^M$ is an observed vector, $\mathbf{A} \in \mathbb{R}^{M\times N}$ is a known matrix, and $\boldsymbol{\varepsilon} \in \mathbb{R}^M$ is noise (Note: the complex version of this problem can also be formulated as Problem 1 essentially via simple $\mathbb{C} \rightleftarrows \mathbb{R}^2$ translation (see Appendix A)).

Problem 1 is a special instance of the mixed integer programming [8], but a direct application of naive solvers for the mixed integer programming leads to exponential computational complexity in N. From a practical viewpoint, continuous optimization approaches [9]–[14] have been utilized for Problem 1 as computationally efficient alternatives. For example, Problem 1 has been tackled with projection of a tentative estimate, say $\mathbf{x}^\diamond \in \mathbb{R}^N$, onto $\mathfrak{D}$ after solving a relaxed continuous optimization problem:

Scheme 1 (A scheme for (1) via regularized least squares)**.**
Step 1:

$$\text{Find } \mathbf{x}^\diamond \in \underset{\mathbf{x}\in\widetilde{\mathfrak{D}}\subset\mathbb{R}^N}{\operatorname{argmin}}\ J_\Theta(\mathbf{x}) := \frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2 + \mu\Theta(\mathbf{x}), \tag{2}$$

where the constraint set $\widetilde{\mathfrak{D}} \supset \mathfrak{D}$ is chosen usually as a connected subset (e.g., the convex hull of $\mathfrak{D}$) of $\mathbb{R}^N$, $\Theta : \mathbb{R}^N \to \mathbb{R}$ is a regularizer, and $\mu > 0$ is a regularization parameter.

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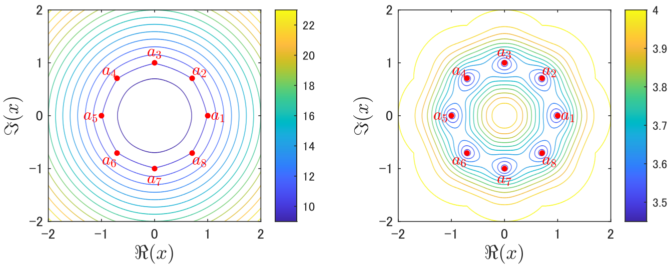

(a) (b)

Fig. 1: Illustrations of the values of Θ in the case where $\mathfrak{D} = \mathfrak{A} := \{a_l := \exp[j(l-1)\pi/4] \mid l = 1, 2, \ldots, 8 =: L\} \subset \mathbb{C} \equiv \mathbb{R}^2$, $\omega_{l,1} = 1/8$ $(l = 1, 2, \ldots, 8)$, and $N = 1$. (a) $\Theta^{\langle 1\rangle}_{\text{SOAV}}(\mathbf{x})$, (b) a proposed nonconvex enhancement Θ_{GME} of $\Theta^{\langle 1\rangle}_{\text{SOAV}}$ by GME (5) with $\mathbf{B}^{\langle l\rangle} = \mathbf{I}$ $(l = 1, 2, \ldots, 8)$.

Step 2: With $\mathbf{x}^\diamond \in \widetilde{\mathfrak{D}} \subset \mathbb{R}^N$, compute the final estimate of $\mathbf{x}^\star \in \mathfrak{D}$ in (1) as

$$\mathbf{x}^\natural = P_{\mathfrak{D}}(\mathbf{x}^\diamond) \in \underset{\mathbf{s}\in\mathfrak{D}}{\operatorname{argmin}}\|\mathbf{s}-\mathbf{x}^\diamond\|_2,$$

where $P_{\mathfrak{D}} : \mathbb{R}^N \to \mathfrak{D} : \mathbf{x} \mapsto P_{\mathfrak{D}}(\mathbf{x})$ is defined to choose randomly one of nearest vectors from $\mathbf{x}$.

In (2), the first term $\frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$ is called a data-fidelity term evaluating the consistency with the linear regression model in (1), while the second term $\mu\Theta : \mathbb{R}^N \to \mathbb{R}$ is called a regularization term designed strategically based on a prior knowledge on $\mathbf{x}^\star$.

For achieving an acceptable estimation of a signal in Problem 1, various prior knowledge, e.g., statistical properties, has been exploited for designing Θ in (2). For example, in a scenario of MIMO signal detection [15], the regularizer Θ in (2) has been found, e.g., as $\Theta = \|\cdot\|_p^p$ $(p = 1, 2)$ [9], [10] and $\Theta = \|\cdot\|_1 \circ D$ [11], where D is a first order difference operator.

For the relaxed continuous constrained optimization problem in (2), it would be desired for the regularizer Θ to penalize any point not in $\mathfrak{D}$. Along this regularization strategy, the so-called SOAV function [13], [14]

$$\Theta^{\langle 1\rangle}_{\text{SOAV}}(\mathbf{x}) := \sum_{l=1}^{L}\|\mathbf{x}-a_l\mathbf{1}\|_{\boldsymbol{\omega}_l,1} := \sum_{l=1}^{L}\sum_{n=1}^{N}\omega_{l,n}|x_n - a_l|, \tag{3}$$

has been used with weighting vectors $\boldsymbol{\omega}_l := [\omega_{l,1}, \omega_{l,2}, \ldots, \omega_{l,N}]^\top \in \mathbb{R}^N_+$ $(l = 1, 2, \ldots, L)$ satisfying $\sum_{l=1}^{L}\omega_{l,n} = 1$ $(n = 1, 2, \ldots, N)$, where

$\|\mathbf{x}\|_{\boldsymbol{\omega}_l,1} := \sum_{n=1}^{N} \omega_{l,n}|x_n|$ stands for the weighted ℓ_1-norm associated with $\boldsymbol{\omega}_l$.

Indeed, the weighted SOAV (W-SOAV) model [14] used (2) with $\Theta = \Theta_{\text{SOAV}}^{\langle 1 \rangle}$ and $\widetilde{D} := \mathbb{R}^N$ for MIMO signal detection. Clearly, in this case, the model (2) is a convex model thanks to the convexity of $\Theta_{\text{SOAV}}^{\langle 1 \rangle}$, and therefore, a solution of (2) can be approximated iteratively by a convex optimization solver [14]. However, Fig. 1 (a) suggests that penalization by $\Theta_{\text{SOAV}}^{\langle 1 \rangle}$ is not contrastive enough for use as Θ in (2) because any point $\mathbf{s}_q \in \mathfrak{D} \subset \mathbb{C}^1$ is never unique minimizer of $\Theta_{\text{SOAV}}^{\langle 1 \rangle}$ over any neighborhood of $\mathbf{s}_q$. More contrastive regularizer than $\Theta_{\text{SOAV}}^{\langle 1 \rangle}$ has been certainly desired for use in (2) of Step 1 of Scheme 1 (see, e.g., Fig. 1 (b) and Fig. 2).

We are interested in the following natural questions:

(Q1) Can we design a class of functions that contains fairly contrastive regularizers Θ for use in (2) ?

(Q2) Can we choose any reasonable function Θ, from such a function class, which is tractable for minimization of J_Θ ?

(Q1) has been examined in special cases, e.g., $\Theta_{\text{SOAV}}^{\langle p \rangle}(\mathbf{x}) := \sum_{l=1}^{L} \|\mathbf{x} - a_l \mathbf{1}\|_{\boldsymbol{\omega}_l,p}$ $(0 \leq p < 1)$ [16] as nonconvex variants of (3). However, any algorithm, of guaranteed to convergence to a global minimizer of $J_{\Theta_{\text{SOAV}}^{\langle p \rangle}}$, has not yet been established mainly because of the severe nonconvexity of $J_{\Theta_{\text{SOAV}}^{\langle p \rangle}}$. This situation tells us that computational tractability of (2) must be considered carefully from the beginning, i.e., (Q1) and (Q2) should be considered simultaneously.

In this paper, we present a positive answer to these questions

i) by designing a function class as

$$\Theta_{\text{GME}}(\mathbf{x}) := \sum_{l=1}^{L} (\|\cdot\|_{\boldsymbol{\omega}_l,1})_{\mathbf{B}^{\langle l \rangle}}(\mathbf{x} - a_l \mathbf{1}), \tag{4}$$

where $(\|\cdot\|_{\boldsymbol{\omega}_l,1})_{\mathbf{B}^{\langle l \rangle}}$ is a nonconvex enhancement (called *Generalized Moreau Enhancement (GME)*) of $\|\cdot\|_{\boldsymbol{\omega}_l,1}$ with a tunable matrix $\mathbf{B}^{\langle l \rangle}$ (see (5)) (Note: Θ_{GME} reproduces $\Theta_{\text{SOAV}}^{\langle 1 \rangle}$ with $\mathbf{B}^{\langle l \rangle} = \mathbf{O}$ (zero matrix));

ii) by exemplifying a fairly contrastive function (see Fig. 1 (b)) in the proposed class of Θ_{GME};

iii) by presenting a choice of $\mathbf{B}^{\langle l \rangle}$ $(l = 1, 2, \ldots, L)$ (see (9) and (10)) which achieves the overall convexity of $J_{\Theta_{\text{GME}}}$;

iv) by proposing an iterative algorithm (see Algorithm 1), based on a relaxation of even symmetric condition [17, Problem 1] required for the so-called seed convex function in cLiGME model (see Section II), with guaranteed to convergence to a global minimizer of $J_{\Theta_{\text{GME}}}$ over $\widetilde{\mathfrak{D}}$ under the overall convexity condition.

Indeed, via numerical experiments in a scenario of MIMO signal detection, we demonstrate the effectiveness of the proposed regularizer Θ_{GME} in the model (2).

We also propose a pair of simple technical improvements for the proposed iterative algorithm in Step 1 of Scheme 1 by exploiting adaptively the discrete information regarding $\mathfrak{D}$. More precisely, we propose (i) to use a strategic perturbation (14) to move the estimate closer to $\mathfrak{D}$ at each iteration $k \in \mathbb{N}$ (this idea is inspired by *superiorization* [18], [19]), and (ii) to update the weights $\omega_{l,n}$ $(l = 1, 2, \ldots, L; n = 1, 2, \ldots, N)$ in (4) adaptively (see (15)) by assigning larger weight to $\omega_{l,n}$ for (l, n) such that the distance between $a_l \in \mathfrak{A}$ and nth coordinate $x_n \in \mathbb{R}$ of the latest estimate $\mathbf{x} \in \mathbb{R}^N$ is smaller (this idea is inspired by iterative reweighting of SOAV [14]). Experimental results demonstrate that these simple techniques improve numerical performance of the proposed Algorithm 1.

Notation. $\mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{R}_{++}$ and $\mathbb{C}$ denote respectively the set of all nonnegative integers, all real numbers, all nonnegative real numbers, all positive real numbers and all complex numbers (j stands for the imaginary unit, and $\Re(\cdot)$ and $\Im(\cdot)$ stand respectively for real and imaginary parts).

Let $\mathcal{H}, \mathcal{K}$ be finite dimensional real Hilbert spaces. The set of all *proper lower semicontinuous convex* functions[1] defined on $\mathcal{H}$ is denoted by $\Gamma_0(\mathcal{H})$. $f \in \Gamma_0(\mathcal{H})$ is said to be *prox-friendly* if $\text{Prox}_{\gamma f} : \mathcal{H} \to \mathcal{H} : x \mapsto \text{argmin}_{y \in \mathcal{H}}[f(y) + \frac{1}{2\gamma}\|y - x\|_{\mathcal{H}}^2]$ is available as a computable operator for any $\gamma \in \mathbb{R}_{++}$. A closed convex set $C \subset \mathcal{H}$ is said to be *simple* if the metric projection $P_C : \mathcal{H} \to \mathcal{H} : x \mapsto \text{argmin}_{y \in \mathcal{H}}\|x - y\|_{\mathcal{H}}$ is available as a computable operator. The set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$. For $\mathfrak{L} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\mathfrak{L}^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ denotes the adjoint operator of $\mathfrak{L}$ (i.e., $(\forall (x, y) \in \mathcal{H} \times \mathcal{K})\ \langle \mathfrak{L}x, y \rangle_{\mathcal{K}} = \langle x, \mathfrak{L}^* y \rangle_{\mathcal{H}}$).

For discussion in Euclidean space, we use boldface letters to express vectors and general font letters to represent scalars. For a matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$, $\mathbf{X}^\top \in \mathbb{R}^{n \times m}$ denotes the transpose of $\mathbf{X}$. The symbols $\mathbf{I}$, $\mathbf{O}$ and $\mathbf{1}$ respectively stand for the identity matrix, the zero matrix and the all one vector.

II. Brief Introduction to cLiGME

Problem 2 (cLiGME model [20], [21]). Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}_l$ $(l = 1, 2, \ldots, L)$, $\widetilde{\mathcal{Z}}_l$ $(l = 1, 2, \ldots, L)$ and $\mathfrak{Z}$ be finite dimensional real Hilbert spaces. Suppose that (a) $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $y \in \mathcal{Y}$ and $\mu > 0$; (b) for each $l = 1, 2, \ldots, L$, $B^{\langle l \rangle} \in \mathcal{B}(\mathcal{Z}_l, \widetilde{\mathcal{Z}}_l)$, $\mathfrak{L}^{\langle l \rangle} \in \mathcal{B}(\mathcal{X}, \mathcal{Z}_l)$ and $\mu_l > 0$; (c) $C (\subset \mathfrak{Z})$ is a nonempty simple closed convex set and $\mathfrak{C} \in \mathcal{B}(\mathcal{X}, \mathfrak{Z})$; (d) $\Psi^{\langle l \rangle} \in \Gamma_0(\mathcal{Z}_l)$ is coercive, $\text{dom}\, \Psi^{\langle l \rangle} = \mathcal{Z}_l$, even symmetry (i.e., $\Psi^{\langle l \rangle} \circ (-\text{Id}) = \Psi^{\langle l \rangle}$), and prox-friendly. Then

i) with a tunable matrix $B^{\langle l \rangle} \in \mathcal{B}(\mathcal{Z}_l \widetilde{\mathcal{Z}}_l)$, the *Generalized Moreau Enhancement (GME)* of $\Psi^{\langle l \rangle}$ is defined by

$$\Psi_{B^{\langle l \rangle}}^{\langle l \rangle}(\cdot) := \Psi^{\langle l \rangle}(\cdot) - \min_{v \in \mathcal{Z}_l} \left[\Psi^{\langle l \rangle}(v) + \frac{1}{2}\|B^{\langle l \rangle}(\cdot - v)\|_{\widetilde{\mathcal{Z}}_l}^2 \right]; \tag{5}$$

ii) the *constrained LiGME (cLiGME)* model is given as

$$\text{Find } x^\diamond \in \underset{\mathfrak{C}x \in C}{\text{argmin}} \frac{1}{2}\|y - Ax\|_{\mathcal{Y}}^2 + \mu \sum_{l=1}^{L} \mu_l \Psi_{B^{\langle l \rangle}}^{\langle l \rangle} \circ \mathfrak{L}^{\langle l \rangle}(x). \tag{6}$$

The regularizer $\Psi_{B^{\langle l \rangle}}^{\langle l \rangle}$ was proposed originally in [17] as an extension of the so-called *GMC penalty* in [22] mainly for the sparsity aware estimation. Furthermore, although $\Psi_{B^{\langle l \rangle}}^{\langle l \rangle}$ with $B^{\langle l \rangle} \neq O$ is nonconvex, the convexity of the cost function in (6) is achieved by a strategic tuning of GME matrices $B^{\langle l \rangle}$ $(l = 1, 2, \ldots, L)$ (see, e.g., [23]).

[1] A function $f : \mathcal{H} \to (-\infty, \infty]$ is (i) proper if $\text{dom}(f) := \{x \in \mathcal{H} \mid f(x) < \infty\} \neq \emptyset$, (ii) lower semicontinuous if $\{x \in \mathcal{H} \mid f(x) \leq \alpha\}$ is closed for $\forall \alpha \in \mathbb{R}$, (iii) convex if $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$ for $\forall x, y \in \mathcal{H}$, $0 < \theta < 1$.

III. Proposed regularizer and algorithm for discrete-valued signal estimation

A. Proposed GME regularizer and cLiGME algorithm

In this section, we propose a class of regularizers $\Theta_{\rm GME}$ in (4), which is designed with the GME functions (5) of $\|\cdot\|_{\boldsymbol{\omega}_l,1}$ $(l=1,2,\ldots,L)$. Indeed, as seen from Fig. 1 (b) (see also Fig. 2), the class of the proposed regularizers $\Theta_{\rm GME}$ contains fairly contrastive functions. Moreover, if $\mathbf{B}^{\langle l\rangle}=\mathbf{O}$ $(l=1,2,\ldots,L)$, then $\Theta_{\rm GME}=\Theta^{\langle 1\rangle}_{\rm SOAV}$ holds. By using $\Theta_{\rm GME}$ (see (4)) in Step 1 of Scheme 1, we propose the following model:

$$\text{Find } \mathbf{x}^{\diamond}\in \underset{\mathbf{x}\in\widetilde{\mathfrak{D}}\subset\mathbb{R}^N}{\operatorname{argmin}} \underbrace{\frac{1}{2}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2+\mu\underbrace{\sum_{l=1}^{L}(\|\cdot\|_{\boldsymbol{\omega}_l,1})_{\mathbf{B}^{\langle l\rangle}}(\mathbf{x}-a_l\mathbf{1})}_{=\Theta_{\rm GME}(\mathbf{x})}}_{=J_{\Theta_{\rm GME}}(\mathbf{x})}, \tag{7}$$

where $\widetilde{\mathfrak{D}}\supset\mathfrak{D}$ is a closed convex set.

In order to solve (7), we consider the following problem which contains (7) as its special instance (see Remark 1).

Problem 3. Let $\mathcal{X}$, $\mathcal{Y}$ and $\widetilde{\mathcal{Z}}_l$ $(l=1,2,\ldots,L)$ be finite dimensional real Hilbert spaces. Suppose that (a) $A\in\mathcal{B}(\mathcal{X},\mathcal{Y})$, $y\in\mathcal{Y}$ and $\mu>0$; (b) for each $l=1,2,\ldots,L$, $B^{\langle l\rangle}\in\mathcal{B}(\mathcal{X},\widetilde{\mathcal{Z}}_l)$, $z^{\langle l\rangle}\in\mathcal{X}$ and $\mu_l>0$; (c) $C(\subset\mathcal{X})$ is a nonempty simple closed convex set; (d) $\Psi^{\langle l\rangle}\in\Gamma_0(\mathcal{X})$ is coercive, $\operatorname{dom}\Psi^{\langle l\rangle}=\mathcal{X}$, even symmetry, and prox-friendly. Then, consider

$$\text{Find } x^{\diamond}\in\underset{x\in C}{\operatorname{argmin}}\underbrace{\frac{1}{2}\|y-Ax\|_{\mathcal{Y}}^2+\mu\sum_{l=1}^{L}\mu_l\Psi^{\langle l\rangle}_{B^{\langle l\rangle}}(x-z^{\langle l\rangle})}_{=:J(x)}, \tag{8}$$

where $\Psi^{\langle l\rangle}_{B^{\langle l\rangle}}:\mathcal{X}\to\mathbb{R}$ is a GME function (5), with a tuning matrix $B^{\langle l\rangle}$, of a convex function $\Psi^{\langle l\rangle}$ $(l=1,2,\ldots,L)$.

Remark 1. The model (7) is a special instance of Problem 3 with $\mathcal{X}=\mathbb{R}^N$, $\mathcal{Y}=\mathbb{R}^M$, $C=\widetilde{\mathfrak{D}}$, $\Psi^{\langle l\rangle}=\|\cdot\|_{\boldsymbol{\omega}_l,1}$ $(l=1,2,\ldots,L)$, $z^{\langle l\rangle}=a_l\mathbf{1}$ $(l=1,2,\ldots,L)$ and $\mu_l=1$ $(l=1,2,\ldots,L)$.

By tuning properly the design parameters of the proposed GME regularizers, we can make the nonconvexly-regularized least squares model convex.

Fact 1 (Overall convexity condition [17] for (8))**.** *Consider Problem 3. Then J in (8) is convex if $(B^{\langle l\rangle})_{l=1}^L$ satisfy*

$$A^*A-\mu\sum_{l=1}^{L}\mu_l{B^{\langle l\rangle}}^*B^{\langle l\rangle} \text{ is positive semidefinite.} \tag{9}$$

For example, the following $B^{\langle l\rangle}$ [22] satisfy (9):

$$B^{\langle l\rangle}:=\sqrt{\frac{\gamma_l}{\mu\mu_l}}A\in\mathcal{B}(\mathcal{X},\mathcal{Y})\quad(l=1,2,\ldots,L), \tag{10}$$

where $\gamma_l\in\mathbb{R}_+$ $(l=1,2,\ldots,L)$ are chosen to satisfy $\sum_{l=1}^L\gamma_l\le 1$.

In a special case where $z^{\langle l\rangle}=0\in\mathcal{X}$ $(l=1,2,\ldots,L)$, the model (8) is reduced to the cLiGME model (6) with $\mathcal{Z}=\mathfrak{Z}=\mathcal{X}$, $\mathfrak{L}^{\langle l\rangle}=\mathrm{Id}$ $(l=1,2,\ldots,L)$ and $\mathfrak{C}=\mathrm{Id}$. For this special case under the condition (9), we can find a global minimizer of (8), by cLiGME algorithm [20], [21].

In the following, we present an iterative algorithm applicable even to general cases $z^{\langle l\rangle}\in\mathcal{X}$ $(l=1,2,\ldots,L)$. The proposed algorithm (12) in Theorem 1 is a variant of cLiGME algorithm [20], [21], [24] (see Remark 2), and Algorithm 1 illustrates a concrete expression of (12) (see also Remark 1).

Theorem 1 (A relaxation of cLiGME algorithm for Problem 3)**.** *Consider Problem 3 under the overall convexity condition* (9)*. Assume* $\operatorname{argmin}_{x\in C}J(x)\neq\emptyset$[2]*. Define the operator* $T:\mathcal{H}:=\mathcal{X}\times(\mathcal{X})^L\times(\mathcal{X})^L\to\mathcal{H}:\big(x,(v^{\langle l\rangle})_{l=1}^L,(w^{\langle l\rangle})_{l=1}^L\big)\mapsto\big(\xi,(\zeta^{\langle l\rangle})_{l=1}^L,(\eta^{\langle l\rangle})_{l=1}^L\big)$ *by*

$$\left\{\begin{array}{l}\xi:=P_C\Big[\big(\mathrm{Id}-\frac{1}{\sigma}(A^*A-\mu\sum_{l=1}^L\mu_l{B^{\langle l\rangle}}^*B^{\langle l\rangle})\big)x\\ \qquad -\frac{\mu}{\sigma}\sum_{l=1}^L\big(\mu_l{B^{\langle l\rangle}}^*B^{\langle l\rangle}v^{\langle l\rangle}+w^{\langle l\rangle}\big)+\frac{1}{\sigma}A^*y\Big],\\ \zeta^{\langle l\rangle}:=z^{\langle l\rangle}+\mathrm{Prox}_{\frac{\mu\mu_l}{\tau}\Psi^{\langle l\rangle}}\Big[\frac{\mu\mu_l}{\tau}{B^{\langle l\rangle}}^*B^{\langle l\rangle}(2\xi-x)\\ \qquad +\big(\mathrm{Id}-\frac{\mu\mu_l}{\tau}{B^{\langle l\rangle}}^*B^{\langle l\rangle}\big)v^{\langle l\rangle}-z^{\langle l\rangle}\Big],\\ \eta^{\langle l\rangle}:=\big(\mathrm{Id}-\mathrm{Prox}_{\mu_l\Psi^{\langle l\rangle}}\big)\big(2\xi-x+w^{\langle l\rangle}-z^{\langle l\rangle}\big),\end{array}\right.$$

where $(\sigma,\tau,\kappa)\in\mathbb{R}_{++}\times\mathbb{R}_{++}\times(1,\infty)$ is chosen to satisfy[3]

$$\left\{\begin{array}{l}(\sigma-\mu L)\mathrm{Id}-\frac{\kappa}{2}A^*A \text{ is positive definite},\\ \tau\ge(\frac{\kappa}{2}+\frac{2}{\kappa})\mu\max\{\mu_l\|B^{\langle l\rangle}\|_{\rm OP}^2\,|\,l=1,2,\ldots,L\}.\end{array}\right. \tag{11}$$

Then, (a) T is an averaged nonexpansive operator[4] *by defining a proper inner product on $\mathcal{H}$ (see, e.g., [17], [21]), and (b) for any initial point $\mathfrak{u}_0\in\mathcal{H}$, the sequence $(\mathfrak{u}_k)_{k\in\mathbb{N}}\subset\mathcal{H}$ with $\mathfrak{u}_k:=\Big(x_k,(v_k^{\langle l\rangle})_{l=1}^L,(w_k^{\langle l\rangle})_{l=1}^L\Big)$ generated by the following* Picard-type fixed point iteration*:*

$$(k\in\mathbb{N})\quad \mathfrak{u}_{k+1}=T(\mathfrak{u}_k) \tag{12}$$

converges to a fixed point $\overline{\mathfrak{u}}=\big(\overline{x},(\overline{v}^{\langle l\rangle})_{l=1}^L,(\overline{w}^{\langle l\rangle})_{l=1}^L\big)\in\operatorname{Fix}(T):=\{\mathfrak{u}\in\mathcal{H}\mid T(\mathfrak{u})=\mathfrak{u}\}\subset\mathcal{H}$*, where $\overline{x}\in\mathcal{X}$ enjoys the condition as a global minimizer $x^{\diamond}\in C$, in* (8)*, of J over C.*

Remark 2. For Problem 3, the condition (9) and Theorem 1 are obtained by the following steps[5].

i) By defining $\widetilde{\Psi}^{\langle l\rangle}:=\Psi^{\langle l\rangle}(\cdot-z^{\langle l\rangle})$ $(l=1,2,\ldots,L)$, we can check that $\widetilde{\Psi}^{\langle l\rangle}_{B^{\langle l\rangle}}=\Psi^{\langle l\rangle}_{B^{\langle l\rangle}}(\cdot-z^{\langle l\rangle})$.

ii) By using $\widetilde{\Psi}^{\langle l\rangle}$, we can express J in (8) equivalently as

$$J(x)=\frac{1}{2}\|y-Ax\|_{\mathcal{Y}}^2+\mu\sum_{l=1}^{L}\mu_l\widetilde{\Psi}^{\langle l\rangle}_{B^{\langle l\rangle}}(x). \tag{13}$$

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[2] $\operatorname{argmin}_{x\in C}J(x)\neq\emptyset$ is guaranteed in many cases, e.g., if C is compact (not limited to this case).

[3] For example, choose $\kappa>1$ and compute (σ,τ) by

$$\left\{\begin{array}{l}\sigma:=\frac{\kappa}{2}\|A\|_{\rm OP}^2+\mu L+(\kappa-1),\\ \tau:=(\frac{\kappa}{2}+\frac{2}{\kappa})\mu\max\{\mu_l\|B^{\langle l\rangle}\|_{\rm OP}^2\mid l=1,2,\ldots,L\}+(\kappa-1),\end{array}\right.$$

where $\|B^{\langle l\rangle}\|_{\rm OP}$ denotes the operator norm of $B^{\langle l\rangle}$ (i.e., $\|B^{\langle l\rangle}\|_{\rm OP}:=\sup_{x\in\mathcal{X},\|x\|_{\mathcal{X}}\le 1}\|B^{\langle l\rangle}x\|_{\widetilde{\mathcal{Z}}_l}$).

[4] An operator $T:\mathcal{X}\to\mathcal{X}$ is said to be *nonexpansive* if $(\forall x,y\in\mathcal{X})$ $\|T(x)-T(y)\|\le\|x-y\|$, in particular, (α-) *averaged nonexpansive* if there exists $\alpha\in(0,1)$ and a nonexpansive operator $R:\mathcal{X}\to\mathcal{X}$ such that $T=(1-\alpha)\mathrm{Id}+\alpha R$.

[5] Even for Problem 2 with a replacement of the seed convex function $\Psi^{\langle l\rangle}$ by its shifted seed convex function $\widetilde{\Psi}^{\langle l\rangle}=\Psi^{\langle l\rangle}(\cdot-z^{\langle l\rangle})$, via an essentialy same discussion, we can derive the overall convexity condition of the objective function and an iterative algorithm with guaranteed convergence to a global minimizer.

Algorithm 1 A relaxation of cLiGME algorithm for (7)

1: Choose $\left(\mathbf{x}_0,(\mathbf{v}_0^{\langle l\rangle})_{l=1}^{L},(\mathbf{w}_0^{\langle l\rangle})_{l=1}^{L}\right)\in\mathbb{R}^N\times(\mathbb{R}^N)^L\times(\mathbb{R}^N)^L$.
2: Choose $(\sigma,\tau,\kappa)\in\mathbb{R}_{++}\times\mathbb{R}_{++}\times(1,\infty)$ satisfying (11).
3: **for** $k=0,1,2,\dots$ **do**
4: { Insert modification in Section III-B if necessary. }
5: $\mathbf{x}_{k+1}\leftarrow P_{\widetilde{\mathfrak{D}}}\big[\big(\mathbf{I}-\frac{1}{\sigma}(\mathbf{A}^\top\mathbf{A}-\mu\sum_{l=1}^{L}\mathbf{B}^{\langle l\rangle\top}\mathbf{B}^{\langle l\rangle})\big)\mathbf{x}_k$ $-\frac{\mu}{\sigma}\sum_{l=1}^{L}(\mathbf{B}^{\langle l\rangle\top}\mathbf{B}^{\langle l\rangle}\mathbf{v}_k^{\langle l\rangle}+\mathbf{w}_k^{\langle l\rangle})+\frac{1}{\sigma}\mathbf{A}^\top\mathbf{y}\big]$
6: **for** $l=1,2,\dots,L$ **do**
7: $\mathbf{v}_{k+1}^{\langle l\rangle}\leftarrow a_l\mathbf{1}+\mathrm{Prox}_{\frac{\mu}{\tau}\|\cdot\|_{\boldsymbol{\omega}_l,1}}\big[\frac{\mu}{\tau}\mathbf{B}^{\langle l\rangle\top}\mathbf{B}^{\langle l\rangle}(2\mathbf{x}_{k+1}-\mathbf{x}_k)$ $+\big(\mathbf{I}-\frac{\mu}{\tau}\mathbf{B}^{\langle l\rangle\top}\mathbf{B}^{\langle l\rangle}\big)\mathbf{v}_k^{\langle l\rangle}-a_l\mathbf{1}\big]$
8: $\mathbf{w}_{k+1}^{\langle l\rangle}\leftarrow\big(\mathrm{Id}-\mathrm{Prox}_{\|\cdot\|_{\boldsymbol{\omega}_l,1}}\big)\big(2\mathbf{x}_{k+1}-\mathbf{x}_k+\mathbf{w}_k^{\langle l\rangle}-a_l\mathbf{1}\big)$
9: **end for**
10: **end for**
11: **return** $\mathbf{x}_{k+1}$

iii) The overall convexity condition (9) for J in (8) is derived by applying [17, Proposition 1] to (13).

iv) Since $\widetilde{\Psi}^{\langle l\rangle}$ is not even symmetry in general, we cannot apply the cLiGME algorithm [20], [21] directly for (13). However, we can relax the even symmetric condition to $z^{\langle l\rangle}$-symmetric condition (i.e., $\widetilde{\Psi}^{\langle l\rangle}(z^{\langle l\rangle}+\cdot)=\widetilde{\Psi}^{\langle l\rangle}(z^{\langle l\rangle}-\cdot)$).

B. Two simple techniques for further improvement

For further improvement of Algorithm 1, we introduce two simple techniques in Algorithm 1 to exploit adaptively the discrete information regarding $\mathfrak{D}$.

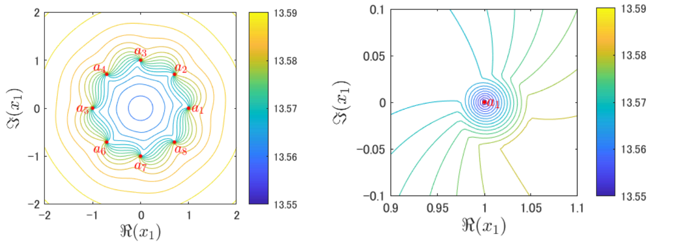
1) Generalized superiorization of cLiGME algorithm: *Superiorization* [18], [19] is known as a technique for an iterative algorithm, e.g., Picard-type fixed point iteration, to reduce a certain objective cost by adding strategic bounded perturbations to updated estimate $\mathbf{x}_k$ $(k\in\mathbb{N})$.

We propose to incorporate a superiorization technique into Algorithm 1 in order to move the estimate closer to $\mathfrak{D}$ at each iteration. More precisely, we use a modification

$$\mathbf{x}_k\leftarrow\mathbf{x}_k+\beta_k\underbrace{(P_{\mathfrak{D}}-\mathrm{Id})(\mathbf{x}_k)}_{=:\mathbf{d}_k}\tag{14}$$

in line 4 of Algorithm 1, where $(\beta_k)_{k\in\mathbb{N}}\subset\mathbb{R}_+$, and $(\mathbf{d}_k)_{k\in\mathbb{N}}\subset\mathbb{R}^N$ is inspired by [19]. The global convergence guarantee is not violated even by the modification (14) if $(\beta_k)_{k\in\mathbb{N}}$ is summable and $(\mathbf{d}_k)_{k\in\mathbb{N}}$ is bounded (see Appendix B2). However, we dare to propose to use more general $(\beta_k)_{k\in\mathbb{N}}\subset\mathbb{R}_+$ which is not necessarily summable. We call such a modification *generalized superiorization*. As will be shown in numerical experiments (see Section IV), the proposed generalized superiorization is effective to guide the sequence $(\mathbf{x}_k)_{k\in\mathbb{N}}$ to the discrete set $\mathfrak{D}$.

2) Iterative reweighting of cLiGME algorithm: The iterative reweighting technique, e.g., [25], has been used to enhance the effectiveness of the regularizer by updating the weights of the regularizer adaptively in an iterative algorithm. Iterative reweighting techniques are also used for Problem 1 [14], [26]. To utilize such a technique in Algorithm 1,



(a) Overall view (b) Enlarged view around a_1

Fig. 2: Illustrations of the values of $\Theta_{\mathrm{GME}}(\mathbf{x})$ in Eq.(4) with $\mathbf{B}^{\langle l\rangle}$ in (17) under the setting of Section IV. For visualization, we set $x_n=0$ $(n=2,3,\dots,50)$.

we propose to set $\omega_{l,n}$ $(l=1,2,\dots,L;\ n=1,2,\dots,N)$ in the seed functions $\|\cdot\|_{\boldsymbol{\omega}_l,1}$ $(l=1,2,\dots,L)$ adaptively by using the latest estimate $\mathbf{x}:=[x_1,x_2,\dots,x_N]^\top$ as [26]

$$\omega_{l,n}=\frac{(|x_n-a_l|+\epsilon)^{-1}}{\sum_{l'=1}^{L}(|x_n-a_{l'}|+\epsilon)^{-1}}.\tag{15}$$

where $\epsilon>0$ is a small number. If $|x_n-a_l|$ is small, then the corresponding $\omega_{l,n}$ becomes large and x_n will be close to a_l. This iterative reweighting method can be realized by inserting

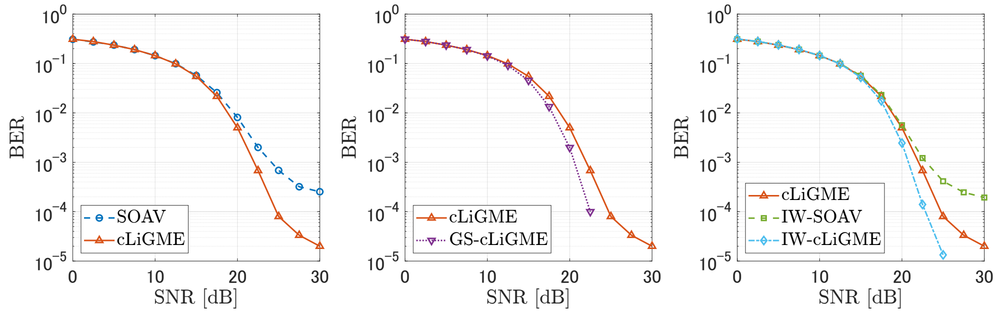
if $k \bmod K == 0$ **then**
Update $\boldsymbol{\omega}_l=[\omega_{l,1},\omega_{l,2},\dots,\omega_{l,N}]^\top$ $(l=1,2,\dots,L)$ as (15) with $\mathbf{x}=\mathbf{x}_k$. (16)
end if

in line 4 of Algorithm 1, where $K\in\mathbb{N}\backslash\{0\}$ controls the frequency of reweighting.

IV. Numerical Experiments

We conducted numerical experiments in a scenario of MIMO signal detection [12] with N-transmit antennas and M-receive antennas $(N=50, M=45)$ in 8PSK (phase shift keying) modulation with constellation set $\mathfrak{A}:=\{a_l:=\exp[j(l-1)\pi/4]\mid l=1,2,\dots,8=:L\}\subset\mathbb{C}$. The task of this experiment is to estimate the transmitted signal $\mathbf{x}^\star\in\mathbb{C}^N$ from the received signal $\mathbf{y}=\mathbf{A}\mathbf{x}^\star+\boldsymbol{\varepsilon}\in\mathbb{C}^M$ with the channel matrix $\mathbf{A}\in\mathbb{C}^{M\times N}$ and a noise $\boldsymbol{\varepsilon}\in\mathbb{C}^M$. In this experiment, we chose randomly (i) $\mathbf{x}^\star\in\mathfrak{D}:=\mathfrak{A}^N$, (ii) $\mathbf{A}:=\sqrt{\mathbf{R}}\mathbf{G}\in\mathbb{C}^{M\times N}$, where each entry of $\mathbf{G}\in\mathbb{C}^{M\times N}$ was sampled from the complex gaussian distribution $\mathcal{CN}(0,1/M)$, and $\mathbf{R}\in\mathbb{R}^{M\times M}$ satisfies $(\mathbf{R})_{i,j}=0.5^{|i-j|}$ $(i=1,2,\dots,M;\ j=1,2,\dots,M)$, and (iii) each entry of $\boldsymbol{\varepsilon}\in\mathbb{C}^M$ was sampled from $\mathcal{CN}(0,\sigma_\varepsilon^2)$ with a variance $\sigma_\varepsilon^2>0$, which was chosen so that $10\log_{10}\frac{1}{\sigma_\varepsilon^2}$ achieved a given SNR (signal-to-noise ratio).

We consider to estimate $\mathbf{x}^\star\in\mathfrak{D}$ with Scheme 1 by employing the convex hull $\mathrm{conv}(\mathfrak{D})$ of $\mathfrak{D}$ as $\widetilde{\mathfrak{D}}$ in (2) via $\mathbb{C}\rightleftarrows\mathbb{R}^2$ translation (see Appendix A). In this experiment, we compared numerical performance of (i) the proposed cLiGME model (7), i.e., the model (2) with $\Theta=\Theta_{\mathrm{GME}}$ in (4), with that of (ii) the SOAV model [14], i.e., the model (2) with $\Theta=\Theta_{\mathrm{SOAV}}^{\langle 1\rangle}$ in (3).



(a) Basic cLiGME (b) Generalized superiorization (c) Iterative reweighting

Fig. 3: BER vs SNR (8PSK, $N = 50$, $M = 45$)

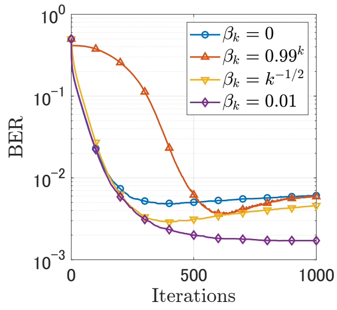


Fig. 4: BER vs iterations

For the cLiGME model, we used Algorithm 1 (denoted by 'cLiGME') by employing the following tuning matrices in (7)

$$\mathbf{B}^{\langle l \rangle} = \sqrt{0.99/\mu L}\widehat{\mathbf{A}} \quad (l = 1, 2, \ldots, L) \tag{17}$$

to achieve the overall convexity condition (9), where μ is a predetermined regularization parameter, and $\widehat{\mathbf{A}}$ is obtained via $\mathbb{C} \rightleftarrows \mathbb{R}^2$ translation (see (18)). Since SOAV model can be reduced to the cLiGME model (7) with $\mathbf{B}^{\langle l \rangle} = \mathbf{O}$ $(l = 1, 2, \ldots, L)$, we used Algorithm 1 (denoted by 'SOAV') with $\mathbf{B}^{\langle l \rangle} = \mathbf{O}$ $(l = 1, 2, \ldots, L)$ for the SOAV model. For both 'cLiGME' and 'SOAV', we employed the same (i) stepsize $(\sigma, \tau, \kappa) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times (1, \infty)$ as footnote 3 in Theorem 1 with $\kappa = 1.001$, and (ii) initial points $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{v}_0^{\langle l \rangle} = \mathbf{0}$ and $\mathbf{w}_0^{\langle l \rangle} = \mathbf{0}$ $(l = 1, 2, \ldots, L)$. Since $\widetilde{\mathfrak{D}}$ is compact, 'cLiGME' and 'SOAV' can find their global minimizers, respectively (see Theorem 1). Algorithm 1 were terminated when the iteration number k exceeded 500.

Before evaluating numerical performance, let us examine contrastiveness of Θ_{GME} used in these experiments. Fig. 2 shows the function values of Θ_{GME} in (4) designed with $\mathbf{B}^{\langle l \rangle}$ in (17) hence achieving the overall convexity condition (9). Each numerical value of $\Theta_{\text{GME}}(\mathbf{x})$ is computed with ISTA-type algorithm [27] (Note: the function value of Θ_{GME} is not required in the proposed Algorithm 1). As seen from Fig. 2 (b), we observe numerically that $\Theta_{\text{GME}}(\mathbf{x})$ is certainly contrastive as a regularizer for discrete-valued signal estimation because each constellation point in $\mathfrak{A}$ corresponds to a local minimizer of $\Theta_{\text{GME}}(\mathbf{x})$ as we expected.

As a performance metric, we adopted averaged BER (bit error rate) over 1,000 independent realizations of $(\mathbf{x}^\star, \mathbf{A}, \boldsymbol{\varepsilon})$. The parameter μ was chosen to achieve the lowest BER from the set $\{10^i \mid i = -10, -9, \ldots, 2\}$ at each SNR.

Fig. 3 (a) shows BER of 'SOAV' and 'cLiGME' at each SNR, where $\omega_{l,n} = 1/8$ $(l = 1, 2, \ldots, 8;\ n = 1, 2, \ldots, N)$ in (7) were fixed. From Fig. 3 (a), 'cLiGME' achieves lower BER than 'SOAV', which implies the effectiveness of the proposed contrastive nonconvex regularizer Θ_{GME} compared with the convex regularizer Θ_{SOAV}.

In the following, we verify the further performance improvements of 'cLiGME' by the proposed (i) generalized superiorization and (ii) iterative reweighting.

To examine the impact of choices of $(\beta_k)_{k \in \mathbb{N}}$ in generalized superiorization (14), we compared generalized superiorization of 'cLiGME' with (i) $\beta_k = 0$ (which reduces to the original 'cLiGME'), (ii) $\beta_k = 0.99^k$ ($(\beta_k)_{k \in \mathbb{N}}$ is summable), (iii) $\beta_k = k^{-1/2}$ ($(\beta_k)_{k \in \mathbb{N}}$ is nonsummable but $\beta_k \to 0$ $(k \to \infty)$), and (iv) $\beta_k = 0.01$. Fig. 4 shows history of BER achieved by generalized superiorization of 'cLiGME' with such $(\beta_k)_{k \in \mathbb{N}}$ in (14), where SNR $= 20$ dB, $\mu = 10^{-4}$ and $\omega_{l,n} = 1/8$ $(l = 1, 2, \ldots, 8;\ n = 1, 2, \ldots, N)$. From Fig. 4, $\beta_k = 0.01$ outperforms the others. Fig. 3 (b) shows BER, at each SNR, of 'cLiGME' and generalized superiorization of 'cLiGME' (denoted by 'GS-cLiGME') with $\beta_k = 0.01$. From Fig. 3 (b), we see 'GS-cLiGME' improves 'cLiGME'.

Fig. 3 (c) shows BER, at each SNR, of (i) 'cLiGME', (ii) iterative reweighting in (16) of 'cLiGME' (denoted by 'IW-cLiGME'), and (iii) iterative reweighting in (16) of 'SOAV' (denoted by 'IW-SOAV'), where the frequency period $K = 100$ in (16) was used (Note: the iterative reweighting of SOAV model was initially proposed [14], [26] with an ADMM-type algorithm). From Fig. 3 (c), 'IW-cLiGME' improves 'cLiGME', while even 'cLiGME' outperforms 'IW-SOAV'.

V. Conclusion

We proposed a class of fairly contrastive regularizers for discrete-valued estimation problems, and presented an iterative algorithm with guaranteed convergence to a global minimizer of the nonconvexly-regularized least squares model. We also proposed two simple techniques for performance improvements. The numerical experiments demonstrate that the proposed model and algorithm have a great potential for challenging discrete-valued signal estimation problem, and that two simple techniques successfully contribute to performance improvements of the proposed algorithm.

Appendix

A. $\mathbb{C} \rightleftarrows \mathbb{R}^2$ *translation*

Consider the complex version of Problem 1 where $\mathfrak{D} (\subset \mathbb{C}^N)$ is a finite set and $(\mathbf{x}^\star, \mathbf{y}, \mathbf{A}, \boldsymbol{\varepsilon}) \in \mathbb{C}^N \times \mathbb{C}^M \times \mathbb{C}^{M \times N} \times \mathbb{C}^M$. The $\mathbb{C} \rightleftarrows \mathbb{R}^2$ translation in this paper should be understood in the following sense:

$$\begin{aligned}
&\widehat{\mathfrak{D}} := \left\{ \begin{bmatrix} \Re(\mathbf{s}) \\ \Im(\mathbf{s}) \end{bmatrix} \in \mathbb{R}^{2N} \,\middle|\, \mathbf{s} \in \mathfrak{D} \right\}, \\
&\widehat{\mathbf{x}}^\star := \begin{bmatrix} \Re(\mathbf{x}^\star) \\ \Im(\mathbf{x}^\star) \end{bmatrix} \in \widehat{\mathfrak{D}} \subset \mathbb{R}^{2N}, \widehat{\mathbf{y}} := \begin{bmatrix} \Re(\mathbf{y}) \\ \Im(\mathbf{y}) \end{bmatrix} \in \mathbb{R}^{2M}, \\
&\widehat{\mathbf{A}} := \begin{bmatrix} \Re(\mathbf{A}) & -\Im(\mathbf{A}) \\ \Im(\mathbf{A}) & \Re(\mathbf{A}) \end{bmatrix} \in \mathbb{R}^{2M \times 2N}, \widehat{\boldsymbol{\varepsilon}} := \begin{bmatrix} \Re(\boldsymbol{\varepsilon}) \\ \Im(\boldsymbol{\varepsilon}) \end{bmatrix} \in \mathbb{R}^{2M}.
\end{aligned} \tag{18}$$

Clearly, via (18), we can translate $\mathbf{y} = \mathbf{A}\mathbf{x}^\star + \boldsymbol{\varepsilon}$ into $\hat{\mathbf{y}} = \hat{\mathbf{A}}\hat{\mathbf{x}}^\star + \hat{\boldsymbol{\varepsilon}}$, and can estimate $\hat{\mathbf{x}}^\star$ by applying Algorithm 1 to the translated real model.

B. Bounded perturbation for Picard-type fixed point iteration

1) Picard iteration: Let $\mathcal{H}$ be a finite dimensional real Hilbert space. Suppose $T : \mathcal{H} \to \mathcal{H}$ is an averaged nonexpansive operator such that $\mathrm{Fix}(T) := \{u \in \mathcal{H} \mid T(u) = u\} \neq \emptyset$. Then, a sequence $(u_k)_{k\in\mathbb{N}}$, generated by the so-called Picard iteration: $u_{k+1} = T(u_k)$ $(k \in \mathbb{N})$ with any initial point $u_0 \in \mathcal{H}$, is guaranteed to converge to a certain fixed point in $\mathrm{Fix}(T)$.

2) Bounded perturbation resilience of Picard iteration [18], [19]: Let $(\beta_k)_{k\in\mathbb{N}}$ be a summable sequence in $\mathbb{R}_+$ and $(d_k)_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$, where such a $(\beta_k d_k)_{k\in\mathbb{N}}$ is said to be a sequence of bounded perturbations. Then, with any initial point $u_0 \subset \mathcal{H}$, $(u_k)_{k\in\mathbb{N}}$ generated by

$$(\forall k \in \mathbb{N})\ u_{k+1} = T(u_k + \beta_k d_k)$$

also converges to a point $\overline{u} \in \mathrm{Fix}(T)$.

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